

# A remark on smooth solutions to a stochastic control problem with a power terminal cost function and stochastic volatilities\*

Yalçın Aktar<sup>†</sup>      Erik Taflin<sup>‡</sup>

August 2013, Version 2014-05-14

Dedicated to the 70th birthday of Ivar Ekeland

## Abstract

Incomplete financial markets are considered, defined by a multi-dimensional non-homogeneous diffusion process, being the direct sum of an Itô process (the price process), and another non-homogeneous diffusion process (the exogenous process, representing exogenous stochastic sources). The drift and the diffusion matrix of the price process are functions of the time, the price process itself and the exogenous process.

In the context of such markets and for power utility functions, it is proved that the stochastic control problem consisting of optimizing the expected utility of the terminal wealth, has a classical solution (i.e.  $C^{1,2}$ ).

This result paves the way to a study of the optimal portfolio problem in incomplete forward variance stochastic volatility models, along the lines of Ekeland et al. [7].

**Key words:** Optimal stochastic control, Smooth solutions, Semilinear parabolic equations, Stochastic volatilities

**MSC 2010:** 49J55 , 35K55, 60H30, 93E20.

## 1 Introduction

The seminal papers [13] and [14], of Merton, on portfolio optimization in continuous time, are formulated for a financial market where the multidimensional (spot) price

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\*The authors thank Nizar Touzi for having drawn their attention to the topic of this article and many constructive discussions.

<sup>†</sup>Chair in Mathematical Finance EISTI and CMAP, Ecole Polytechnique Paris, [yar@eisti.eu](mailto:yar@eisti.eu)

<sup>‡</sup>Chair in Mathematical Finance EISTI and AGM, Université de Cergy, [taflin@eisti.fr](mailto:taflin@eisti.fr)

is a Markov process. In particular, when the price is a non-homogeneous diffusion, the Hamilton-Jacobi-Bellman equation was derived, and it was solved explicitly in the special case of a log-normal price process and a power utility function. Important generalizations of Merton's work have been accomplished by the study of the regularity of viscosity solutions to the HJB equation and by the use of duality methods.

The purpose of this article is to solve the optimal portfolio problem, in the context of the incomplete markets models **(FM)** defined below, with "stochastic volatility", by first establishing that the HJB equation has a classical solution and then applying a verification theorem:

- **Financial Market (FM)**

There is one tradeable risk-free asset with vanishing interest rate. There are exactly  $n$  tradeable basic risky assets in the market and  $X$  is a  $n$ -dim. Itô process, whose coordinates are (a simple function of) the spot prices of the tradeable basic risky assets. Exogenous stochastic sources are represented by a  $d$ -dim. non-homogeneous diffusion process  $Y$  with the diffusion matrix being invertible. The process  $(X, Y)$  is a non-homogeneous diffusion process, so the drift and the diffusion matrix of  $X$  are functions of the time  $t$ ,  $X_t$  (the price process at  $t$ ) and  $Y_t$  (the exogenous process at  $t$ )

The above optimal portfolio problem will only be considered for power utility functions, defining the bequest, without consumption and in its unconstrained version, in the sense that the portfolio is allowed to take any value in  $\mathbb{R}^{1+n}$ . It will be referenced by **(OPP)**. The comments below concerning related papers, only refer to this unconstrained case.

In the case of power utility functions, generalizations of Merton's framework to incomplete markets with stochastic volatility, not necessarily of the above type **(FM)**, by proving that the HJB equation has a classical solution, have been studied by many authors under various hypotheses, cf. [18], [15], [12], [6], [1] and references therein. In these works, the coefficients of the SDEs defining the model are independent of price  $X_t$ .

The markets models in [18], [15], [1] are of the type **(FM)**. In [18] the pure investment problem (i.e. only a power bequest function and no consumption) is considered,  $n = d = 1$ , which permits to transform the HJB equation into a linear PDE, by a fractional substitution and obtain explicit solutions. In [15] the pure investment problem is considered,  $n$  and  $d$  are arbitrary, the coefficients of the model are independent of time and by an exponential substitution the HJB equation is transformed into a semi-linear PDE, which is proved to have a classical solution. A necessary and sufficient condition is given by [15, Remark 3.1, Eq. (3.7)] for the existence of fractional substitution transforming the HJB equation into a linear equation, like in [18]. Reference [1] considers the problem with consumption and bequest function for general  $n \geq d$  and the coefficients of the model are restricted to satisfy (after a simple transformation) the condition [15, Remark 3.1, Eq. (3.7)], here with time dependent coefficients. A fractional substitution then transforms

the HJB equation into a semi-linear equation, which is proved to have a classical solution.

In the works [12], with general  $n$ , and [6], with  $n = 1$ , the exogenous process is of Ornstein-Uhlenbeck type, driven by a subordinator with càdlàg sample paths, i.e. a Lévy process with a.s. non-decreasing sample paths. The drifts and volatilities are time-independent. Reference [5] is based on duality methods.

Our main motivation, for this work, is to find a solution of the portfolio problem **(OPP)**, useful for solving corresponding optimal portfolio problems in the framework of incomplete forward variance stochastic volatility models. Such models have a natural formulation in an infinite dimensional setting close to what is used for Zero-Coupon Bond markets and the dynamics of forward rate curves, cf. for theoretical developments [3], [4], [7], [8] and for applications [2] and references therein. This requires an extension of earlier works on the portfolio problem **(OPP)** to markets of the type **(FM)**, where the coefficients of the models are allowed to be functions also of time and asset prices.

We have accomplished this, first by extending the framework of [15] to cover the case of market models of the type **(FM)**, and then by solving the portfolio problem **(OPP)** in this context. Although our problem is more complex, the main ideas of [15] can be adapted to our proofs. Also in our case a standard substitution (see (3.3)) transforms the HJB equation into a semilinear second order PDE (3.4), quadratic in the first derivatives. This PDE is regularized by, decreasing the quadratic growth to a linear growth of the Hamiltonian in its first order derivative variable. This corresponds to a multiplication of the Legendre-Fenchel transformed Hamiltonian by a cut-off function. The existence and uniqueness (see Lemma 3.2) of a solution to the regularized HJB equation (3.22) follows from a standard result [9, Theorem 6.2 Chap VI]. After a reformulation in terms of a stochastic control problem, a crucial estimate, uniform in the cut-off, of the derivative of regularized solution is obtained, Lemma 3.3. The proof of this lemma is based on Appendix A, which generalizes [10, Lemma 11.4] to our case. The uniform estimate of the derivative permits to prove the convergence of the regularized solution to a solution of the semilinear PDE, when the cut-off “disappears”, Theorem 3.1. A verification result is then proved using elementary properties of the Girsanov transformation, which gives the main result Theorem 2.4.

Certain differences in the hypothesis of this work and [15] are due to that some growth conditions of model coefficients, announced as linear growth in [15] should be replaced by square root growth (see point 4. of Remark 2.3).

We finish this long introduction by setting some of the notations to be used.

#### **Notations:**

For linear spaces  $F$  and  $G$ ,  $L(F, G)$  is the linear space of linear continuous operators of  $F$  into  $G$ .  $L(F, G)$  is endowed with the operator norm.

A linear operator and its matrix representation (w.r.t. a given orthonormal basis) will not be distinguished.  $A'$  denotes the adjoint operator of a linear operator  $A$  and  $|A|$  the operator norm of  $A$ .

Let  $n \geq 1$  be an integer and for  $i \in \{0, \dots, n\}$  let  $\mathcal{O}_i$  be open subsets of finite

dimensional vector spaces. For a function  $f : \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \rightarrow \mathcal{O}_0$ , when well-defined, the partial derivative of order  $m$  w.r.t. variables  $x_1, \dots, x_m$ , where for every  $i$ ,  $x_i \in \mathcal{O}_j$  for some  $j$ , is denoted  $f_{x_1 \dots x_m}$ . To avoid confusions, occasionally we write  $\nabla_x f$  instead of  $f_x$  etc.

If not stated otherwise:  $z = (x, y), r = (p, q) \in E = \mathbb{R}^n \times \mathbb{R}^d$ .

The identity function is denoted  $I$ , possibly with an index indicating in which set.

Scalar product of  $a, b \in F$ , depending on the context,  $a \cdot b$ ,  $(a, b)_F$  and  $(a, b)$  are used.

The open ball of radius  $R$  centered at 0 in a normed space  $E$  is denoted  $B_E(0, R)$  and  $\bar{B}_E(0, R)$  is its closure (or just  $B(0, R)$  and  $\bar{B}(0, R)$ ).

## 2 The mathematical model and main result

Let  $B$  and  $W$  be two independent standard Brownian motions, of dimension  $m$  and  $d$  respectively, restricted to a time interval  $[0, T]$  (with  $T > 0$ ), on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  endowed with the complete filtration  $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  generated by  $\tilde{W} = (B, W)$ .

We consider, on the time interval  $[0, T]$ , a financial market with  $n$  risky assets, whose price processes  $S^i$ ,  $1 \leq i \leq n$  are strictly positive, and one risk-free asset whose interest rate is 0. Since  $S^i$  is strictly positive, we can express the dynamics in terms of the  $\mathbb{R}^n$ -valued process  $X$ , whose  $i$ :th coordinate is  $X^i = \ln(S^i)$ . The process  $X$  is supposed to satisfy the SDE

$$dX_t = \tilde{\mu}_1(t, Z_t) dt + \sigma_1(t, Z_t) dB_t + \sigma_2(t, Z_t) dW_t, \quad X_0 \in \mathbb{R}^n, \quad (2.1)$$

where  $Z_t = (X_t, Y_t)$  is an  $E = \mathbb{R}^n \times \mathbb{R}^d$  valued process and  $Y$  is a  $\mathbb{R}^d$ -valued process representing the “exogenous stochastic factors” of the model and is supposed to satisfy the SDE

$$dY_t = \mu_2(t, Y_t) dt + dW_t, \quad Y_0 \in \mathbb{R}^d. \quad (2.2)$$

In equation (2.1),  $\tilde{\mu}_1$ ,  $\sigma_1$  and  $\sigma_2$  are continuous functions of  $[0, T] \times E$  into  $\mathbb{R}^n$ ,  $L(\mathbb{R}^m, \mathbb{R}^n)$  and  $L(\mathbb{R}^d, \mathbb{R}^n)$  respectively. In equation (2.2),  $\mu_2$  is a continuous function of  $[0, T] \times \mathbb{R}^d$  into  $\mathbb{R}^d$ .

We introduce the volatility functions  $\sigma : [0, T] \times E \rightarrow L(\mathbb{R}^{m+d}, \mathbb{R}^n)$  of the SDE (2.1) and  $\Sigma : [0, T] \times E \rightarrow L(\mathbb{R}^{m+d}, E)$  of the system of SDEs (2.1) and (2.2) by, for all  $t \in [0, T]$ ,  $z \in E$ ,  $a = (b, c)$ ,  $b \in \mathbb{R}^m$  and  $c \in \mathbb{R}^d$

$$\sigma(t, z)a = \sigma_1(t, z)b + \sigma_2(t, z)c \text{ and } \Sigma(t, z)a = (\sigma(t, z)a, c). \quad (2.3)$$

We also introduce the functions  $\sigma^i : [0, T] \times E \rightarrow \mathbb{R}^{m+d}$ ,  $1 \leq i \leq n$ , by

$$\sigma^i(t, z) = (\sigma^{i1}(t, z), \dots, \sigma^{i(m+d)}(t, z))$$

and the drift function  $\mu_1 : [0, T] \times E \rightarrow \mathbb{R}^n$  by

$$\mu_1 = \tilde{\mu}_1 + \beta, \text{ where } \beta = (\beta^1, \dots, \beta^n), \beta^i(t, z) = \frac{1}{2} |\sigma^i(t, z)|^2,$$

and the norm is the Euclidean norm in  $E$ .

The coefficients in (2.1) and (2.2) are supposed to satisfy various conditions for different purposes.

Conditions related to existence of  $Z$ :

**Condition A.**

$A_1)$  If  $f$  is the function  $\tilde{\mu}_1 : [0, T] \times E \rightarrow \mathbb{R}^n$  or  $\sigma : [0, T] \times E \rightarrow L(\mathbb{R}^{m+d}, \mathbb{R}^n)$  then

$$f \in C^1([0, T] \times E) \text{ and } \exists C \text{ s.t. } \forall (t, z) \in [0, T] \times E \quad |f_z(t, z)| \leq C. \quad (2.4)$$

$A_2)$  The function  $\mu_2$  satisfies:

$$\mu_2 \in C^1([0, T] \times \mathbb{R}^d; \mathbb{R}^d) \text{ and } \exists C \text{ s.t. } \forall (t, y) \in [0, T] \times \mathbb{R}^d \quad |\nabla_y \mu_2(t, y)| \leq C. \quad (2.5)$$

Conditions related to existence of a  $C^{1,2}$  solution of the HJB equation:

**Condition B.**

$B_1)$   $C^{1,2}$  regularity of  $\sigma$ ,

$$\sigma \in C^{1,2}([0, T] \times E, L(\mathbb{R}^{m+d}, \mathbb{R}^n)), \quad (2.6)$$

$B_2)$  For  $(t, z) \in [0, T] \times E$  and for  $i = 1, 2$ , let  $M_i(t, z) = \sigma_i(t, z)\sigma_i(t, z)'$ ,  $M(t, z) = M_1(t, z) + M_2(t, z)$ . We shall consider the following conditions:

$$\exists C \in \mathbb{R} \quad : \quad \forall (t, z) \in [0, T] \times E, \quad |\sigma(t, z)| \leq C, \quad (2.7)$$

$$\forall (t, z) \in [0, T] \times E, \quad M_1(t, z)^{-1} \text{ exists and } \exists C \in \mathbb{R} \quad : \quad \forall (t, z), \quad |M_1(t, z)^{-1}| \leq C, \quad (2.8)$$

$B_3)$  The functions  $\tilde{\mu}_1$  and  $\mu_2$  satisfy the following square-root growth condition:  
There exists  $C \in \mathbb{R}$  such that for all  $(t, z) \in [0, T] \times E$

$$|\tilde{\mu}_1(t, z)| \leq C\sqrt{1 + |y|} \text{ and } |\mu_2(t, y)| \leq C\sqrt{1 + |y|}. \quad (2.9)$$

$B_4)$  For all  $(t, z) \in [0, T] \times E$ ,  $M(t, z)$  is invertible and if  $f(t, z) = |M(t, z)^{-1/2}\mu_1(t, z)|^2$  then  $f$  satisfies

$$f \in C^{0,1}([0, T] \times E) \text{ and } \exists C \text{ s.t. } \forall (t, z) \in [0, T] \times E \quad |f(t, z)| + |f_z(t, z)| \leq C. \quad (2.10)$$

Condition related to the application of a verification theorem:

**Condition C.**

$C_1)$  For all  $(t, z) \in [0, T] \times E$ ,  $M(t, z)$  is invertible and if  $f(t, z) = \sigma_2(t, z)'M(t, z)^{-1}\mu_1(t, z)$ , then  $f$  satisfies

$$f \in C^{0,1}([0, T] \times E) \text{ and } \exists C \text{ s.t. } \forall (t, z) \in [0, T] \times E \quad |f_z(t, z)| \leq C\sqrt{1 + |y|}. \quad (2.11)$$

It is standard that the system of SDE (2.1) and (2.2) has a unique strong solution, when  $A_1$ ) and  $A_2$ ) of Condition A are satisfied, cf. [17].

When  $M(t, z)$  is invertible, we shall use the notation

$$N(t, z) = \sigma_2'(t, z)M(t, z)^{-1}\sigma_2(t, z). \quad (2.12)$$

One immediately obtains the following result, where point 1 permits to simplify equation (2.1) (see Remark 2.2).

**Lemma 2.1.**

1. Suppose that  $\sigma$  satisfies the conditions (2.4), (2.6), (2.7) and (2.8). Then these conditions are also satisfied with  $\sigma$  replaced by  $\tilde{\sigma} = (M_1^{1/2}, \sigma_2)$
2. If  $\sigma$  satisfies conditions (2.7) and (2.8), then

$$\exists C \in (0, 1) : \forall (t, z) \in [0, T] \times E, \quad 1 - |N(t, z)| \geq C. \quad (2.13)$$

**Proof:**

1. According to (2.7) and (2.8) there exist  $0 < c < C$  such that the spectrum of  $M_1(t, z)$  is a subset of  $(c, C)$  for all  $(t, z)$ . Denote by  $\mathbb{C}_+$  the set of complex numbers with real part  $\geq 0$ . Let  $\gamma$  be a simple positively oriented continuous closed curve in  $\mathring{\mathbb{C}}_+$ , the interior of  $\mathbb{C}_+$ , enclosing  $[c, C]$ . The square root function is holomorphic in  $\mathring{\mathbb{C}}_+$ , so by the Dunford-Taylor integral,  $\forall (t, z) \in [0, T] \times E$ ,

$$M_1(t, z)^{1/2} = -\frac{1}{2\pi i} \int_{\gamma} \zeta^{1/2} R(M_1(t, z), \zeta) d\zeta, \quad (2.14)$$

where  $R(M_1(t, z), \zeta) = (M_1(t, z) - \zeta I)^{-1}$  is the the resolvent of  $M_1(t, z)$  at  $\zeta$ . Using that the derivative w.r.t.  $l$  being one of the variables  $t$  and  $z_i$  is given by

$$\frac{\partial}{\partial l} R(M_1(t, z), \zeta) = -R(M_1(t, z), \zeta) \left( \frac{\partial}{\partial l} M_1(t, z) \right) R(M_1(t, z), \zeta),$$

one verifies that the conditions (2.4), (2.6), (2.7) and (2.8) are satisfied by  $\tilde{\sigma} = (M_1^{1/2}, \sigma_2)$ .

2. According to (2.8) there exists  $c > 0$  such that for all  $(t, z) \in [0, T] \times E$  and  $a \in \mathbb{R}^n$  we have  $(a, M(t, z)a) \geq c|a|^2 + |\sigma_2(t, z)'a|^2$  or equivalently  $|a|^2 \geq c|M(t, z)^{-1/2}a|^2 + |\sigma_2(t, z)'M(t, z)^{-1/2}a|^2$ . By (2.6) it then exists  $C > 0$  such that  $|\sigma_2(t, z)'M(t, z)^{-1/2}|^2 \leq 1 - c/|M(t, z)| \leq 1 - c/C$  for all such  $(t, z)$ . Then by the definition of  $N$ ,  $|N(t, z)| = |M_1(t, z)^{-1/2}\sigma_2(t, z)|^2 = |\sigma_2(t, z)'M(t, z)^{-1/2}|^2 \leq 1 - c/C$ .  $\square$

**Remark 2.2.** We will use that, according to 1. of Lemma 2.1, the volatility function  $\sigma_1$  in (2.1) can be replaced by  $M_1^{1/2}$ , when the function  $M_1^{-1}$  exists, i.e.

$$dX_t = \tilde{\mu}_1(t, Z_t) dt + M_1(t, Z_t)^{1/2} d\tilde{B}_t + \sigma_2(t, Z_t) dW_t, \quad X_0 \in \mathbb{R}^n, \quad (2.1')$$

where

$$\tilde{B}_t = \int_0^t M_1(s, Z_s)^{-1/2} \sigma_1(t, Z_s) dB_s.$$

In fact, by Lévy's characterization theorem  $(\tilde{B}, W)$  is a standard  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  B.m. and one establish that:

If conditions A<sub>1</sub>) and A<sub>2</sub>) are satisfied and  $M^{-1}(t, z)$  exists for all  $(t, z) \in [0, T] \times E$ , then  $Z$  is a strong solution to the system (2.1) and (2.2) iff  $Z$  is a strong solution to the system (2.1') and (2.2).

**Remark 2.3.**

1. The the volatility function  $\Sigma$  in (2.3) has for all  $(t, z) \in [0, T] \times E$  the property:  $\Sigma(t, z)$  is on-to  $E$  iff  $\sigma_1(t, z)$  is on-to, or equivalently

$$\Sigma(t, z) \text{ is onto } E \text{ iff } M_1(t, z) = \sigma_1(t, z)\sigma_1(t, z)' \text{ is invertible.}$$

When  $M_1(t, z)$  is invertible, one has

$$|(\Sigma(t, z)\Sigma(t, z)')^{-1/2}| \leq 1 + |M_1(t, z)^{-1/2}| + |\sigma_2(t, z)'M_1(t, z)^{-1/2}|.$$

In particular  $\Sigma$  satisfies the uniform parabolicity condition iff the right hand side of this inequality is uniformly bounded on  $[0, T] \times E$ .

2. At several occasions, [9, Theorem 6.2, p169] will be used to prove important intermediary results of this paper and in particular for the existence of solutions of eq. (3.22). A literal application requires  $m = n$ . However, if  $\Sigma$  is supposed to satisfy the uniform parabolicity condition, the case  $m > n$  can be reduced to the case  $m = n$  by a redefinition of the B.m., as in Remark 2.2.
3. Let  $M(t, z)$  be invertible and let the linear operator  $A(t, z)$  be as in (3.13). One easily establish that if one of the linear operators  $M_1(t, z)$ ,  $I - N(t, z)$  or  $A(t, z)$  has an inverse then all three are invertible.
4. The optimal portfolio problem for a power utility function, in case of coefficients independent of time and price in the system of SDE (2.1) and (2.2), was treated in [15] (in particular see [15, (H3a), p.66]). For this case there is an important difference between our hypotheses and those of [15]. In fact, (2.9) of B<sub>2</sub>) imposes a square-root growth condition on  $\mu_2$ , but the corresponding assumption (H3ai) of [15] only imposes a linear growth condition on  $\mu_2$ . However, the proof of [15, Lemma 4.1] is wrong under the linear growth condition (H3ai) and it turns out that our square-root growth condition is exactly what is needed to make it correct.

It was noted in [15, Remark 2.2] that, when  $\mu_2$  satisfies the Lipschitz condition (2.5) of Condition A, then  $|Y|$  satisfies the following linear growth condition,  $\exists C \in \mathbb{R}$  such that  $\forall t \in [0, T]$

$$|Y_t| \leq C \left( 1 + \int_0^t |W_u| du + |W_t| \right), \quad (2.15)$$



and consequently that there exists  $\varepsilon > 0$  such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\varepsilon |Y_t|^2} \right] < +\infty. \quad (2.16)$$

Consider now an agent with a power utility function  $U$  and with a self-financing investment policy  $\pi$  in the above financial market, whose purpose is to optimize the expected utility  $\mathbb{E}[U(\mathfrak{W}_T)]$  of the final wealth  $\mathfrak{W}_T$ . More precisely, suppose that the liquidation value  $\mathfrak{W}_t > 0$ , at time  $t \in [0, T]$ , satisfies

$$d\mathfrak{W}_t = \mathfrak{W}_t \sum_{1 \leq i \leq n} \pi_t^i \frac{1}{S_t^i} dS_t^i,$$

where  $\pi = (\pi_t^1, \dots, \pi_t^n)$  is an adapted  $\mathbb{R}^n$ -valued process. Consequently,  $\pi_t^i$  is the fraction of  $\mathfrak{W}_t$  held in asset nr.  $i$ ,  $1 \leq i \leq n$ . The set  $\mathcal{A}$  of admissible controls is here (following [15, formula (2.5)]) the set of all  $\mathbb{F}$  progressively measurable  $\mathbb{R}^n$ -valued processes  $\pi$  satisfying

$$\exists \varepsilon > 0 \text{ such that } \sup_{0 \leq t \leq T} \mathbb{E} \left[ e^{\varepsilon |\sigma(Z_t)' \pi_t|^2} \right] < \infty. \quad (2.17)$$

For a given power  $a \in (-\infty, 1) \setminus \{0\}$ , the utility function  $U$  is defined by

$$U(\mathfrak{w}) = \frac{\mathfrak{w}^a}{a}, \quad \mathfrak{w} > 0. \quad (2.18)$$

The gain function  $J$  is defined on  $[0, T] \times (0, \infty) \times E \times \mathcal{A}$  by

$$J(t, \mathfrak{w}, z, \pi) = \mathbb{E}[U(\mathfrak{W}_T) | \mathfrak{W}_t = \mathfrak{w}, Z_t = z], \quad (2.19)$$

where the process  $Z = (X, Y)$  satisfies the SDEs (2.1) and (2.2) and the process  $\mathfrak{W}$  satisfies the controlled SDE

$$d\mathfrak{W}_s = \mathfrak{W}_s (\pi_s' \mu_1(s, Z_s) ds + \pi_s' \sigma(s, Z_s) d\tilde{W}_s), \quad \mathfrak{W}_t = \mathfrak{w}, \quad t \leq s \leq T. \quad (2.20)$$

The agent's optimization problem is then formulated as the following stochastic optimal control problem, with value function  $v$ :  $\forall (t, \mathfrak{w}, z) \in [0, T] \times (0, \infty) \times E$

$$v(t, \mathfrak{w}, z) = \sup_{\pi \in \mathcal{A}} J(t, \mathfrak{w}, z, \pi). \quad (2.21)$$

Here, as in the seminal work of Merton, if a solution  $v$  of equation (2.21) exists, then the (positive) homogeneity of the utility function  $U$  and the linearity of equations (2.20) in the variable  $\mathfrak{W}$  gives directly that the function  $\mathfrak{w} \mapsto v(t, \mathfrak{w}, z)$  must be homogeneous of degree  $a$ , i.e.

$$\forall \mathfrak{w} > 0, \quad v(t, \mathfrak{w}, z) = \mathfrak{w}^a v(t, 1, z). \quad (2.22)$$

We note (cf. [15, Remark 1.2]) that if  $f : [0, T] \times E \rightarrow \mathbb{R}^n$  is Borel measurable and if there is  $C \in \mathbb{R}$ , such that for all  $t \in [0, T]$  and  $z = (x, y) \in E$  one has  $|\sigma(t, z)' f(t, z)| \leq C(1 + |y|)$ , then the control  $\pi \in \mathcal{A}$ , when  $\pi_t = f(t, Z_t)$ . This is obtained by (2.16).

Finally we state here the main result of this paper, which is a corollary of Proposition 4.1 and to be proved in the following sections:



**Theorem 2.4.** *If Condition A, Condition B and Condition C are satisfied, then*

1. *the value function  $v$ , defined by formula (2.21), satisfies (2.22) and  $v(\cdot, 1, \cdot) \in C^{1,2}([0, T] \times E)$ ,*
2. *there is a unique optimal control  $\hat{\pi} \in \mathcal{A}$ .*

### 3 The semi-linear HJB Equation

The Hamilton-Jacobi-Bellman equation for the stochastic control problem (2.21) reads: For  $z = (x, y)$  and all  $(t, \mathfrak{w}, z) \in [0, T] \times (0, \infty) \times E$

$$\begin{aligned} & v_t(t, \mathfrak{w}, z) + \tilde{\mu}_1(t, z)'v_x(t, \mathfrak{w}, z) + \mu_2(t, y)'v_y(t, \mathfrak{w}, z) + \frac{1}{2}\text{Tr}(\sigma(t, z)'v_{xx}(t, \mathfrak{w}, z)\sigma(t, z)) \\ & + \text{Tr}(\sigma_2(t, z)'v_{xy}(t, \mathfrak{w}, z)) + \frac{1}{2}\Delta_y v(t, \mathfrak{w}, z) + \sup_{\pi \in \mathbb{R}^n} (\pi' \mu_1(t, z) \mathfrak{w} v_{\mathfrak{w}}(t, \mathfrak{w}, z) \\ & + \frac{1}{2}|\pi' \sigma(t, z)|^2 \mathfrak{w}^2 v_{\mathfrak{w}\mathfrak{w}}(t, \mathfrak{w}, z) + \pi' \sigma(t, z) \sigma(t, z)' \mathfrak{w} v_{\mathfrak{w}x}(t, \mathfrak{w}, z) + \pi' \sigma_2(t, z) \mathfrak{w} v_{\mathfrak{w}y}(t, \mathfrak{w}, z)) = 0, \end{aligned} \quad (3.1)$$

and

$$v(T, \mathfrak{w}, z) = \frac{\mathfrak{w}^a}{a}. \quad (3.2)$$

As usually, the supposed homogeneity property (2.22) leads to the following ansatz:

$$v(t, \mathfrak{w}, z) = \frac{\mathfrak{w}^a}{a} e^{-u(t, z)}, \text{ for } (t, \mathfrak{w}, z) \in [0, T] \times (0, \infty) \times E. \quad (3.3)$$

Under the hypothesis of this ansatz, we obtain that

$$\left( \mathfrak{w} \frac{\partial}{\partial \mathfrak{w}} \right)^n v(t, \mathfrak{w}, z) = a^n v(t, \mathfrak{w}, z), \text{ for } n \in \mathbb{N}.$$

Insertion of this and (3.3) into equation (3.1) gives, ignoring the argument  $(t, z)$  in  $u(t, z)$ : For  $z = (x, y)$  and all  $(t, z) \in [0, T] \times E$

$$-u_t - \frac{1}{2}\text{Tr}(\sigma(t, z)'u_{xx}\sigma(t, z)) - \text{Tr}(\sigma_2(t, z)'u_{xy}) - \frac{1}{2}\Delta_y u + H(t, z, u_z) = 0 \quad (3.4)$$

and

$$u(T, z) = 0, \quad (3.5)$$

where the function  $[0, T] \times E \times E \ni (t, (x, y), (p, q)) = (t, z, r) \mapsto H(t, z, r) \in \mathbb{R}$  is defined by

$$\begin{aligned} H(t, z, r) = & \frac{1}{2}|\sigma(t, z)'p|^2 + q'\sigma_2(t, z)'p + \frac{1}{2}|q|^2 - \tilde{\mu}_1(t, z)'p - \mu_2(t, z)'q \\ & + a \max_{\pi \in \mathbb{R}^n} \left( \pi'(\mu_1(t, z) - \sigma(t, z)\sigma(t, z)'p - \sigma_2(t, z)q) - \frac{1-a}{2}|\sigma(t, z)'\pi|^2 \right). \end{aligned} \quad (3.6)$$

Since the PDE (3.4) is linear in the second order derivatives, it is by definition semilinear.

Under the condition (2.8),  $M(t, z)$  is invertible for all  $(t, z) \in [0, T] \times E$ , so the unique solution  $\tilde{\pi}$  to the maximization problem in formula (3.6) is explicitly given by

$$\tilde{\pi}(t, z, r) = \frac{1}{1-a} M(t, z)^{-1} (\mu_1(t, z) - M(t, z)p - \sigma_2(t, z)q). \quad (3.7)$$

Substitution into (3.6) gives, for all  $(t, z) \in [0, T] \times E$  the second degree polynomial  $H(t, z, r)$  in  $r$ :

$$\begin{aligned} H(t, z, r) &= \frac{1}{2} \frac{1}{1-a} |\sigma(t, z)'p|^2 + \frac{1}{1-a} q' \sigma_2(t, z)'p + \frac{1}{2} q' \left( I_d + \frac{a}{1-a} \sigma_2(t, z)' M(t, z)^{-1} \sigma_2(t, z) \right) q \\ &+ p' \left( \beta(t, z) - \frac{1}{1-a} \mu_1(t, z) \right) - q' \left( \mu_2(t, z) + \frac{a}{1-a} \sigma_2(t, z)' M(t, z)^{-1} \mu_1(t, z) \right) \\ &+ \frac{1}{2} \frac{a}{1-a} \mu_1(t, z)' M(t, z)^{-1} \mu_1(t, z). \end{aligned} \quad (3.8)$$

For later reference we note that

$$\begin{aligned} &|\sigma(t, z)' \tilde{\pi}(t, z, r)|^2 \\ &= (1-a)^{-2} (\mu_1(t, z) - M(t, z)p - \sigma_2(t, z)q)' M(t, z)^{-1} (\mu_1(t, z) - M(t, z)p - \sigma_2(t, z)q). \end{aligned} \quad (3.9)$$

In order to study the existence of classical solutions to the semilinear equation (3.4) we introduce the linear space by  $C_l^{1,2}$  of all functions  $u \in C^0([0, T] \times E) \cap C^{1,2}([0, T] \times E)$  satisfying the following growth condition: With  $z = (x, y)$ , there exists  $C \in \mathbb{R}$  such that

$$\forall (t, z) \in [0, T] \times E, |M(t, z)^{1/2} u_x(t, z)| + |u_y(t, z)| \leq C(1 + |y|). \quad (3.10)$$

The remaining part of this section is devoted to prove the existence result, of a classical solution to the semilinear HJB equation, formulated by

**Theorem 3.1.** *Assume that Condition A, Condition B and Condition C are satisfied Then there exists a solution  $u \in C_l^{1,2}$  to the semilinear equation (3.4) with the terminal condition (3.5).*

In order to prove Theorem 3.1, we reformulate the semilinear equation (3.4), with the terminal condition (3.5), as a stochastic control problem.

Supposing (2.8), we can re-write for all  $(t, z) \in [0, T] \times E$ , the convex second degree polynomial  $E \ni r \mapsto H(t, z, r)$ , given by (3.8), on the following form:

$$H(t, z, r) = \frac{1}{2} (r, A(t, z)r) - (r, l(t, z)) + k(t, z), \quad (3.11)$$

where, with  $r = (p, q)$ ,

$$N(t, z) = \sigma_2(t, z)' M(t, z)^{-1} \sigma_2(t, z), \quad (3.12)$$

$$A(t, z)r = \left( \frac{1}{1-a}M(t, z)p + \frac{1}{1-a}\sigma_2(t, z)q, \frac{1}{1-a}\sigma_2(t, z)'p + q + \frac{a}{1-a}N(t, z)q \right), \quad (3.13)$$

$$\ell(t, z) = (\ell_1(t, z), \ell_2(t, z)) = \left( \frac{1}{1-a}\mu_1(t, z) - \beta(t, z), \mu_2(t, y) + \frac{a}{1-a}\sigma_2(t, z)'M(t, z)^{-1}\mu_1(t, z) \right), \quad (3.14)$$

$$k(t, z) = \frac{1}{2} \frac{a}{1-a} \mu_1(t, z)' M(t, z)^{-1} \mu_1(t, z). \quad (3.15)$$

One checks that the symmetric positive linear operator  $A(t, z)$  is positive definite when (2.8) is satisfied.

For all  $(t, z) \in [0, T) \times E$ , the convex function  $L(t, z, \cdot)$  on  $E$  is defined by the Legendre-Fenchel transformation (in  $-\bar{r}$ ) of the function  $\bar{r} \mapsto H(t, z, \bar{r})$ :

$$L(t, z, r) = \sup_{\bar{r} \in E} (-(\bar{r}, r) - H(t, z, \bar{r})). \quad (3.16)$$

When  $A(t, z)$  is positive definite, which is supposed in the sequel by imposing (2.8), then  $L(t, z, r)$  have the following explicit form:

$$L(t, z, r) = \frac{1}{2} (r - \ell(t, z))' A(t, z)^{-1} (r - \ell(t, z)) - k(t, z). \quad (3.17)$$

By convexity

$$H(t, z, r) = \sup_{\bar{r} \in E} (-(\bar{r}, r) - L(t, z, \bar{r})), \quad (3.18)$$

where the supremum is realized for  $\bar{r} = \hat{r}(t, z, r)$ ,

$$\hat{r}(t, z, r) = \ell(t, z) - A(t, z)r. \quad (3.19)$$

A stochastic control problem, corresponding to the semilinear equation (3.4), with the terminal condition (3.5), is now

$$u(t, z) = \inf_{\nu \in \mathcal{U}_t} \mathbb{E} \left[ \int_t^T L(s, Z_s, \nu_s) ds \middle| Z_t = z \right], \quad (3.20)$$

where  $\mathcal{U}_t$  is the set of all square integrable progressively measurable  $E$ -valued process independent of  $\mathcal{F}_t$ .

For  $R > 0$ , a regularization  $H^R(t, z, r)$  of the Hamiltonian  $H(t, z, r)$  at  $(t, z, r) \in [0, T] \times E^2$  is defined by

$$H^R(t, z, r) = \sup_{|\bar{r}| \leq R} (-(\bar{r}, r) - L(t, z, \bar{r})). \quad (3.21)$$

We consider then the HJB equation with the Hamiltonian  $H$  replaced by  $H^R$ ,

$$\begin{aligned} -u_t^R - \frac{1}{2} \text{Tr}(\sigma(t, z)' u_{xx}^R \sigma(t, z)) - \text{Tr}(\sigma_2(t, z)' u_{xy}^R) - \frac{1}{2} \Delta_y u^R + H^R(t, z, u_z^R) &= 0, \\ \text{with final data } u^R(T, \cdot) &= 0. \end{aligned} \quad (3.22)$$

We shall see that, [9, Theorem 6.2 Chap VI] permits to deduce the existence of a  $C^{1,2}$  solution  $u^R$ .

We shall use simple estimates for  $A(t, z)$  and  $A(t, z)^{-1}$  (when it exists) and their first derivatives. The expression (3.13) gives directly

$$|A(t, z)| \leq 1 + \frac{2}{1-a}(|M(t, z)| + |N(t, z)|), \quad (t, z) \in [0, T] \times E. \quad (3.23)$$

and

$$|A_z(t, z)| \leq \frac{2}{1-a}(1 + |M(t, z)^{1/2}| + 2|M(t, z)^{-1/2}|)|\sigma_z(t, z)|, \quad (t, z) \in [0, T] \times E. \quad (3.24)$$

The following explicit expression of  $A(t, z)$  (cf. Schur complement) is convenient for estimating  $A(t, z)^{-1}$ :

$$A(t, z) = T(t, z)D(t, z)T(t, z)', \quad (3.25)$$

where

$$T(t, z)r = (p, \sigma_2(t, z)'M(t, z)^{-1}p + q) \text{ and } D(t, z)r = \left(\frac{1}{1-a}M(t, z)p, q - N(t, z)q\right). \quad (3.26)$$

Since

$$|D(t, z)^{-1}| \leq (1-a)|M(t, z)^{-1}| + |(I - N(t, z))^{-1}| \leq (1-a)|M(t, z)^{-1}| + \frac{1}{1 - |N(t, z)|}$$

and

$$|(T(t, z))^{-1}| \leq 1 + |M(t, z)^{-1/2}|$$

it follows that

$$|A(t, z)^{-1}(t, z)| \leq 2 \left( (1-a)|M(t, z)^{-1}| + \frac{1}{1 - |N(t, z)|} \right) (1 + |M(t, z)^{-1}|). \quad (3.27)$$

A direct calculation gives that

$$|N_z(t, z)| \leq 4|M(t, z)^{-1/2}||\sigma_z(t, z)|. \quad (3.28)$$

Inequalities (3.24), (3.27) and (3.28) give

$$|\nabla_z A(t, z)^{-1}(t, z)| \leq \frac{32}{1-a} \left( 1 + |M(t, z)^{1/2}| + |M(t, z)^{-1}| + \frac{1}{1 - |N(t, z)|} \right)^5 |\sigma_z(t, z)|. \quad (3.29)$$

**Lemma 3.2.** *Let (2.4) and (2.5) of Condition A be satisfied and let  $\sigma$  satisfy (2.6), (2.7), (2.8) and (2.13), then equation (3.22) has a solution  $u^R \in C^{1,2}([0, T] \times E) \cap C^0([0, T] \times E)$  being unique in the subset of such functions with polynomial growth.*

*Proof:* Let the assumptions of the lemma be satisfied. Then, by Remark 2.2 and Lemma 2.1, we can suppose without restriction that  $m = n$  and that  $\sigma$  is invertible. The existence and uniqueness  $u^R$  then follows from [9, Theorem 6.2 Chap VI]. In fact the hypothesis of this theorem are satisfied:

1. The functions  $\mu_1, \tilde{\mu}_1, \beta, \mu_2$  and  $\sigma$  are  $C^1$  with bounded derivative according to (2.4) and (2.7). It then follows that there exists  $C \in \mathbb{R}$  such that for all  $z = (x, y) \in E$  and  $f \in \{\tilde{\mu}_1, \mu_1, \beta, \sigma\}$

$$|f(t, z)| \leq C(1+|z|), \quad |f_z(t, z)| \leq C, \quad |\mu_2(t, y)| \leq C(1+|y|) \text{ and } |\nabla_y \mu_2(t, y)| \leq C. \quad (3.30)$$

2. By (3.30), (2.6) and (2.7),  $[0, T] \times E \ni (t, z) \mapsto \Sigma(t, z)$  is  $C^2$  and bounded together with its first derivative. Then by (2.8),  $\Sigma(t, z)$  is the invertible and  $E \ni z \mapsto \Sigma(t, z)^{-1}$  is bounded together with its first derivative. So with the definition  $\theta(t, z, r) = \Sigma(t, z)^{-1}r$ , points a) and b) of assumption (6.9) of [9, Theorem 6.2 Chap VI] are satisfied.
3. By condition (2.4) and conditions (2.7), (2.8) and (2.13), inequalities (3.27) and (3.29) give for some  $C \in \mathbb{R}$

$$|A(t, z)^{-1}(t, z)| \leq C \text{ and } |\nabla_z A(t, z)^{-1}(t, z)| \leq C|\sigma_z(t, z)| \leq CC_1, \quad (3.31)$$

where the last inequality follows from (3.30) for some  $C_1$  independent of  $z$ .

According to inequality (3.31) there exists  $C > 0$  such that for all  $(t, z, r) \in [0, T] \times E \times E$

$$|L(t, z, r)| \leq C|r - \ell(t, z)|^2 + |k(t, z)|, \quad (3.32)$$

$$|L_z(t, z, r)| \leq C|r - \ell(t, z)|^2 + C|\ell_z(t, z)||r - \ell(t, z)| + |k_z(t, z)|. \quad (3.33)$$

By (3.30), (2.8) and (2.13) there is  $C > 0$  such that for all  $(t, z) \in [0, T] \times E$

$$|k(t, z)| + |l(t, z)|^2 \leq C(1 + |z|)^2. \quad (3.34)$$

Using now (3.30), (3.28), (2.7), (2.8), (2.13) it follows that the  $C$  in (3.34) can be chosen such that

$$|k_z(t, z)| + |l_z(t, z)|^2 \leq C(1 + |z|)^2. \quad (3.35)$$

The above estimates for  $L, l$  and  $k$  and their first derivatives imply that the  $C^1$  function  $[0, T] \times E \times \bar{B}_E(0, R) \ni (t, z, r) \mapsto L(t, z, r)$  is bounded together with its first derivative by a second degree polynomial in  $z$ . This shows that also point c) of assumption (6.9) of [9, Theorem 6.2 Chap VI] is satisfied. Therefore all the assumptions of [9, Theorem 6.2 Chap VI] are true.

□

We have the following estimate, uniform in  $R$ , for the derivative  $u_z^R$ .

**Lemma 3.3.** *Let the conditions (2.9), (2.10) and (2.11) and the conditions of Lemma 3.2 be satisfied. Then there exists  $C \in \mathbb{R}$  such that for all  $(t, z) = (t, (x, y)) \in [0, T] \times E$  and  $R > 0$*

$$|u_z^R(t, z)| \leq C(1 + |y|).$$

*Proof:* The solution  $u^R$  to (3.22) has a stochastic control representation, which solution is obtained by verification, cf. [9, Theorem 6.4 Chap VI] and [17, Th ??]

$$u^R(t, z) = \inf_{\nu \in \mathcal{U}_t(R)} \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T L(s, \bar{Z}_s, \nu_s) ds \middle| \bar{Z}_t = z \right], \quad (3.36)$$

where  $\mathcal{U}_t(R) = \{\nu \in \mathcal{U}_t : |\nu| \leq R \text{ a.e. } dt d\mathbb{Q}\}$  and where the controlled dynamics of  $\bar{Z}$  is given by

$$d\bar{Z}_t = \nu_t dt + \Sigma(\bar{Z}_t) d\bar{W}_t^{\mathbb{Q}} \quad (3.37)$$

with  $\bar{W}^{\mathbb{Q}}$  a  $(m + d)$ -dimensional standard Brownian motion under  $\mathbb{Q}$  and  $\bar{W}^{\mathbb{Q}} = (B^{\mathbb{Q}}, W^{\mathbb{Q}})$ ,  $W^{\mathbb{Q}}$  being  $d$ -dimensional.

Let  $\hat{Z}$  be the solution of (3.37) with the control  $\nu_t = \hat{r}_R(\bar{Z}_t, u_z^R(t, \bar{Z}_t))$ , where the optimal control function  $\hat{r}_R$  is given by (3.19). Then the optimal control  $\hat{\nu}$  for (3.36) is given by  $\hat{\nu}_t = \hat{r}_R(\hat{Z}_t, u_z^R(t, \hat{Z}_t))$  and by (3.36),

$$u^R(t, z) = \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T L(\hat{Z}_s, \hat{\nu}_s) ds \middle| \hat{Z}_t = z \right]. \quad (3.38)$$

By the controlled SDE (3.37) and results from [10], the derivative of  $u^R$  is given by (see Appendix Corollary A.2 for a proof):

$$u_z^R(t, z) = \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T L_z(\hat{Z}_s, \hat{\nu}_s) ds \middle| \hat{Z}_t = z \right]. \quad (3.39)$$

The function  $k_z$  is a bounded by (2.10). The functions  $\nabla\mu_1$ ,  $\nabla\mu_2$  and  $\nabla\beta$  are bounded by (3.30) and the function  $\nabla(\sigma_2' M^{-1} \mu_1)$  is bounded by (2.11). This shows that the function  $\ell_z$  is bounded. It now follows from inequality (3.33), with a new constant  $C$  and for some positive constant  $C_1$ , that for all  $(t, z, r) \in [0, T] \times E \times E$

$$|L_z(t, z, r)| \leq C_1 |r - \ell(t, z)|^2 + C_1 |\ell_z(t, z)| |r - \ell(t, z)| + |k_z(t, z)| \leq C(1 + |r - \ell(t, z)|^2). \quad (3.40)$$

By conditions (2.7) and (2.13) and by inequality (3.23) the function  $[0, T] \times E \ni (t, z) \mapsto |A(t, z)|$  is bounded by a constant  $C > 0$ , so  $|A(t, z)^{-1}| \geq C^{-1}$ . Hence with a new constant  $C$

$$|L_z(t, z, r)| \leq C \left( 1 + \frac{1}{2} (r - \ell(t, z), A(t, z)^{-1} (r - \ell(t, z))) \right) = C(1 + L(t, z, r) + k(t, z)).$$

The function  $k$  is bounded according to (2.10). Let  $c(k)$  be a bound. Then, for all  $(t, z, r) \in E \times E$ ,

$$|L_z(t, z, r)| \leq C(1 + L(t, z, r) + c(k)). \quad (3.41)$$

Let  $\bar{Z}^0$  be a solution of the SDE (3.37) with  $\nu = 0$ . Inequality (3.41), the stochastic control representations (3.36) and (3.39) and inequality (3.32) then give,

for some positive constants  $C_1$  independent of  $(t, z)$  and  $R$ , that

$$\begin{aligned}
|u_z^R(t, z)| &= \left| \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T L_z(s, \hat{Z}_s, \hat{\nu}_s) ds \middle| \hat{Z}_t = z \right] \right| \\
&\leq \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T C \left( 1 + L(s, \hat{Z}_s, \hat{\nu}_s) + c(k) \right) ds \middle| \hat{Z}_t = z \right] \\
&\leq C \left( C_1 + \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T L(s, \hat{Z}_s, \hat{\nu}_s) ds \middle| \hat{Z}_t = z \right] \right) \\
&\leq C \left( C_1 + \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T L(s, \bar{Z}_s^0, 0) ds \middle| \bar{Z}_t^0 = z \right] \right).
\end{aligned} \tag{3.42}$$

Due to conditions (2.9) and (2.11), there is  $C \in \mathbb{R}$  such that for all  $(t, z) = (t, (x, y)) \in [0, T] \times E$ ,

$$|\ell(t, z)|^2 \leq C(1 + |y|). \tag{3.43}$$

The estimate (3.32) and  $|k(t, z)| \leq c(k)$  then give, with a new constant  $C$ , that for all  $(t, z) \in [0, T] \times E$ ,

$$|L(t, z, 0)| \leq C(1 + |y|).$$

It now follows from (3.42) that, for some constant  $C_2$  independent of  $(t, z)$  and  $R$

$$|u_z^R(t, z)| \leq C \left( C_1 + C_2 \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T (|y| + |W_s^{\mathbb{Q}}|) ds \right] \right).$$

This proves the existence of  $C \in \mathbb{R}$  such that for all  $(t, z, R) \in [0, T] \times E \times (0, \infty)$ ,

$$|u_z^R(t, z)| \leq C(1 + |y|), \text{ where } z = (x, y). \tag{3.44}$$

□

*Proof of Theorem 3.1.* The function  $E \ni r \rightarrow -(r, u_z^R(t, z))_E - L(t, z, r)$  (with  $L$  as in (3.16)) attains its maximum on  $E$  for

$$\hat{r}^R(t, z) = \ell(t, z) - A(t, z)u_z^R(t, z).$$

Recall that  $A(t, z)$  is bounded by (3.23) and  $\ell$  satisfies (3.43). Moreover, by Lemma 3.3 there is a  $C$  such that, for all  $(t, z, R) \in [0, T] \times E \times (0, \infty)$ ,  $u_z^R$  satisfies the inequality  $|u_z^R(t, z)| \leq C(1 + |y|)$ . Hence for given  $c \geq 0$ , there exists a positive constant  $C_1$  independent of  $R$  such that

$$|\hat{r}^R(t, z)| \leq C_1, \quad \forall (t, z) \in [0, T] \times E \text{ such that } |y| \leq c.$$

Therefore, for  $R \geq C_1$ , we have

$$\begin{aligned}
H^R(t, z, u^R(t, z)) &= \sup_{\bar{r} \in \bar{B}_E(0, R)} [-\bar{r}' u^R(t, z) - L(t, z, \bar{r})] \\
&= \sup_{\bar{r} \in E} [-\bar{r}' u^R(t, z) - L(t, z, \bar{r})] \\
&= H(t, z, u^R(t, z)),
\end{aligned} \tag{3.45}$$

for all  $(t, z) \in [0, T] \times E$  such that  $|y| \leq c$ . Since  $c$  can be chosen arbitrarily large, this implies that  $u^R$  is a  $C^{1,2}$  solution to (3.4)-(3.5) satisfying (3.10).

□



## 4 Existence of a classical solution to the stochastic optimal control problem

This section is devoted to proving the main result of this note, Theorem 2.4, which is a trivial corollary (not stated formally) of the following Proposition 4.1 and of the existence of a classical solution in Theorem 3.1. In Proposition 4.1 a verification result, which relates a solution of the semilinear equation (3.4), with terminal condition (3.5), to the original stochastic control problem (2.21), is proved using elementary properties of the Girsanov transformation (cf. [15]).

**Proposition 4.1.** *If  $u \in C_l^{1,2}$  is a solution of the semilinear PDE (3.4), with the terminal condition (3.5), if the assumptions (2.5) and (2.4) are satisfied and if*

$$[0, T] \times E \ni (t, z) = (t, (x, y)) \mapsto |M(t, z)^{-1/2} \mu_1(t, z)| (1 + |y|)^{-1} \text{ is a bounded function} \quad (4.1)$$

*then the value function of (2.21) is given by*

$$v(t, \mathfrak{w}, z) = \frac{\mathfrak{w}^a}{a} e^{-u(t, z)}, \text{ for } (t, \mathfrak{w}, z) \in [0, T] \times (0, \infty) \times E.$$

*There is a unique optimal control process  $\hat{\pi} \in \mathcal{A}$ ,*

$$\hat{\pi}_t = \tilde{\pi}(t, Z_t, u_z(t, Z_t)),$$

*where  $\tilde{\pi}$  is defined by (3.7).*

*Proof:*

1. We first prove that  $\hat{\pi} \in \mathcal{A}$ . With norms in relevant linear spaces, it follows from (3.9) that

$$\begin{aligned} |\sigma(t, z)' \tilde{\pi}(t, z, r)|^2 &= \frac{1}{(1-a)^2} |M(t, z)^{-1/2} (\mu_1(t, z) - M(t, z)p - \sigma_2(t, z)q)|^2 \\ &\leq \frac{3}{(1-a)^2} (|M(t, z)^{-1/2} \mu_1(t, z)|^2 + |M(t, z)^{1/2} p|^2 + |M(t, z)^{-1/2} \sigma_2(t, z)q|^2). \end{aligned} \quad (4.2)$$

Using conditions (4.1) and (3.10), this gives for some  $C \in \mathbb{R}$  that

$$\begin{aligned} |\sigma(t, z)' \hat{\pi}(t, z, u_x(t, z), u_y(t, z))|^2 &\leq \frac{3}{(1-a)^2} (|M(t, z)^{-1/2} \mu_1(t, z)|^2 + |M(t, z)^{1/2} u_x(t, z)|^2 + |M(t, z)^{-1/2} \sigma_2(t, z) u_y(t, z)|^2) \\ &\leq C(1 + |y|^2). \end{aligned} \quad (4.3)$$

It now follows from (2.16) that  $\hat{\pi}$  satisfies (2.17), so  $\hat{\pi} \in \mathcal{A}$ .

2. Let  $\pi \in \mathcal{A}$  be an admissible control. We define the measure  $\mathbb{Q}^\pi$  by

$$\frac{d\mathbb{Q}^\pi}{d\mathbb{P}} = \exp \left( \int_0^T a \pi'_s \sigma(s, Z_s) d\tilde{W}_s - \frac{1}{2} \int_0^T |a \sigma(s, Z_s)' \pi_s|^2 ds \right). \quad (4.4)$$

Since  $\pi \in \mathcal{A}$ , condition (2.17) implies that  $\mathbb{Q}^\pi$  is a probability measure (cf. Exercise 1.40, §1, Ch.VIII [16]). The  $m$  and  $d$ -dimensional processes  $B^\pi$  and  $W^\pi$  respectively and the  $m + d$ -dimensional process  $\tilde{W}^\pi = (B^\pi, W^\pi)$ , are defined by

$$d\tilde{W}_t^\pi = d\tilde{W}_t - a\sigma(t, Z_t)' \pi_t dt, \quad \tilde{W}_0^\pi = 0.$$

According to Girsanov's theorem these three processes are standard multi-dimensional  $\mathbb{Q}^\pi$ -B.m. and

$$\begin{aligned} dX_t &= (\tilde{\mu}_1(t, Z_t) + a\sigma(t, Z_t)\sigma(t, Z_t)' \pi_t) dt + \sigma(t, Z_t) d\tilde{W}_t^\pi, \\ dY_t &= (\mu_2(t, Y_t) + a\sigma_2(t, Z_t)' \pi_t) dt + dW_t^\pi, \end{aligned} \quad (4.5)$$

3. If  $u \in C_l^{1,2}$  is a solution to (3.4) and (3.5) and  $\pi \in \mathcal{A}$ , then an exponential  $\mathbb{Q}^\pi$ -local martingale  $\xi^\pi$ , is defined by

$$\begin{aligned} \xi_t^\pi &= \exp \left( - \int_0^t (u_x(s, Z_s)' \sigma_1(s, Z_s) dB_s^\pi + (u_y(s, Z_s)' + u_x(s, Z_s)' \sigma_2(s, Z_s)) dW_s^\pi \right) \\ &\quad - \frac{1}{2} \int_0^t (|\sigma_1(s, Z_s)' u_x(s, Z_s)|^2 + |u_y(s, Z_s) + \sigma_2(s, Z_s)' u_x(s, Z_s)|^2) ds \Big). \end{aligned} \quad (4.6)$$

The particular case  $\xi^{\hat{\pi}}$  is a  $\mathbb{Q}^{\hat{\pi}}$ -martingale. In fact, according to inequality (3.10) there exists  $C \in \mathbb{R}$  such that for all  $t \in [0, T]$

$$|u_x(t, Z_t)' \sigma_1(t, Z_t)|^2 + |u_y(t, Z_t)' + u_x(t, Z_t)' \sigma_2(t, Z_t)|^2 \leq C(1 + |Y_t|^2).$$

The second SDE in (4.5), with  $\pi = \hat{\pi}$ , and inequality (4.3) then imply that the above constant  $C$  can be chosen such that

$$|Y_t| \leq |Y_0| + \int_0^t (|\mu_2(s, Y_s)| + a|\sigma_2(s, Z_s)' \hat{\pi}_s|) ds + |W_t^{\hat{\pi}}| \leq |Y_0| + C \int_0^t (1 + |Y_s|) ds + |W_t^{\hat{\pi}}|,$$

which using Grönwall's inequality leads to the existence of  $\varepsilon > 0$  such that

$$\sup_{0 \leq t \leq T} \mathbb{E}^{\mathbb{Q}^{\hat{\pi}}} \left[ e^{\varepsilon |Y_t|^2} \right] < \infty.$$

The claim now follows by using the above mentioned exercise.

4. By (2.20)

$$\mathfrak{W}_T = \mathfrak{w} \exp \left( \int_t^T (\pi_s' \mu_1(s, Z_s) ds + \pi_s' \sigma(s, Z_s) d\tilde{W}_s) - \frac{1}{2} \int_t^T |\pi_s' \sigma(s, Z_s)|^2 ds \right),$$

which together with (2.19) gives

$$J(t, \mathfrak{w}, z, \pi) = \frac{\mathfrak{w}^a}{a} \mathbb{E}^{\mathbb{Q}^\pi} \left[ \exp \left( \int_t^T l(s, Z_s, \pi_s) ds \right) \middle| Z_t = z \right], \quad (4.7)$$

where for  $\alpha \in \mathbb{R}^n$ ,  $l(t, z, \alpha) = a\alpha' \mu_1(t, z) - (a(1 - a)/2) |\sigma(t, z)' \alpha|^2$ .

Now let  $u \in C_l^{1,2}$  be a solution to (3.4) and (3.5). By formula (4.5), Itô's formula applied to  $u(t, Z_t)$  gives:

$$\begin{aligned}
u(T, Z_T) &= u(t, Z_t) + \int_t^T \left( u_t(s, Z_s) + u_x(s, Z_s)'(\tilde{\mu}_1(s, Z_s) + a\sigma(s, Z_s)\sigma(s, Z_s)'\pi_s) \right. \\
&\quad + u_y(s, Z_s)'(\mu_2(s, Y_s) + a\sigma_2(s, Z_s)'\pi_s) + \frac{1}{2}\text{Tr}(\sigma(s, Z_s)'u_{xx}(s, Z_s)\sigma(s, Z_s)) \\
&\quad \left. + \frac{1}{2}\Delta_y u(s, Z_s) + \text{Tr}(\sigma_2(s, Z_s)'u_{xy}(s, Z_s)) \right) ds \\
&\quad + \int_t^T \left( u_x(s, Z_s)'\sigma_1(s, Z_s)dB_s^\pi + (u_y(s, Z_s)' + u_x(s, Z_s)'\sigma_2(s, Z_s))dW_s^\pi \right) \geq u(t, Z_t) \\
&\quad + \int_t^T l(s, Z_s, \pi_s)ds + \int_t^T \left( u_x(s, Z_s)'\sigma_1(s, Z_s)dB_s^\pi + (u_y(s, Z_s)' + u_x(s, Z_s)'\sigma_2(s, Z_s))dW_s^\pi \right) \\
&\quad + \frac{1}{2} \int_t^T \left( |\sigma_1(s, Z_s)'u_x(s, Z_s)|^2 + |u_y(s, Z_s) + \sigma_2(s, Z_s)'u_x(s, Z_s)|^2 \right) ds.
\end{aligned} \tag{4.8}$$

$\xi^\pi$  is a  $\mathbb{Q}^\pi$ -supermartingale and the terminal condition  $u(T, z) = 0$ , so it follows from (4.8) that,

$$\mathbb{E}^{\mathbb{Q}^\pi} \left[ \exp \left( \int_t^T l(s, Z_s, \pi_s) ds \right) \middle| \mathcal{F}_t \right] \leq \exp(-u(t, Z_t)) \mathbb{E}^{\mathbb{Q}^\pi} \left[ \frac{\xi_T^\pi}{\xi_t^\pi} \middle| \mathcal{F}_t \right] \leq \exp(-u(t, Z_t)), \tag{4.9}$$

Since  $\pi \in \mathcal{A}$  was arbitrary up to now, it follows from (4.7) that

$$\forall \pi \in \mathcal{A}, \quad J(t, x, z, \pi) \leq \frac{x^a}{a} \exp(-u(t, z)). \tag{4.10}$$

Now choosing in particular  $\pi = \hat{\pi}$ , there is equality in (4.8). Since  $\xi^{\hat{\pi}}$  is a  $\mathbb{Q}^{\hat{\pi}}$ -martingale, it then follows that all the inequalities in (4.9) and (4.10) are equalities, which proves that

$$v(t, x, z) = \sup_{\pi \in \mathcal{A}} J(t, x, z, \pi) = J(t, x, z, \hat{\pi}) = \frac{x^a}{a} e^{-u(t, z)}.$$

□

## A Appendix

Let  $E = \mathbb{R}^p$ ,  $F = \mathbb{R}^q$  where  $p, q \in \mathbb{N}^*$ .  $W$  is a  $q$ -dimensional standard Brownian motion on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$  endowed with the complete filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  generated by  $W$ . For  $t \in [0, T]$ , where  $T > 0$  is fixed, we denote by  $\mathcal{U}_t^2$ , the collection of all progressively measurable  $E$ -valued processes  $\phi$  independent of  $\mathcal{F}_t$ , such that  $\mathbb{E} \left[ \int_t^T |\phi(s)|^2 ds \right] < \infty$ . For  $R > 0$ , we set

$$\mathcal{U}_t^2(R) = \{\nu \in \mathcal{U}_t^2 : |\nu| \leq R \text{ a.e. } dt d\mathbb{Q}\}.$$

Given  $(t, z) \in [0, T] \times E$ , we consider the following SDE for the process  $Z^{t,z}$ :

$$Z^{t,z}(s) = z + \int_t^s \nu(u) du + \int_t^s \sigma(u, Z^{t,z}(u)) dW_u, \quad t \leq s \leq T, \quad (\text{A.1})$$

where  $[0, T] \times E \ni (t, z) \mapsto \sigma(t, z) \in L(F, E)$  is a continuous function, Lipschitz continuous in  $z$  (with Lipschitz constant  $K$  independent of  $t$  and where the control process  $\nu(\cdot) \in \mathcal{U}_t^2(R)$ ).

By classical theorems (cf. [11, 9. Theorem, p.83 and 10. Corollary, p.85]), there exist a strong solution and a constant  $C(q, K)$  such that for all  $t \in [0, T]$ ,  $z, z' \in E$  and  $q \geq 2$

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} |Z^{t,z}(s)|^q \right] \leq C(q, K) C'(1 + |z|^q), \quad (\text{A.2})$$

where  $C' = 1 + R^q + \sup_{t \leq s \leq T} |\sigma(s, 0)|$ , and

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} |Z^{t,z'}(s) - Z^{t,z}(s)|^q \right] \leq C(q, K) |z - z'|^q. \quad (\text{A.3})$$

Denote by  $C^{0,1,0}([0, T] \times E \times E)$  the linear space of all real functions on  $[0, T] \times E \times E$  such that  $f$  and  $f_z$  are continuous, where  $f_z(t, z, v) = \nabla_z f(t, z, v)$ . Let  $L : [0, T] \times E \times E \rightarrow \mathbb{R}$  be a function satisfying the following conditions:

- (a)  $L \in C^{0,1,0}([0, T] \times E \times E)$
  - (b)  $\exists k, C_1 \geq 0$  such that  $|L(t, z, v)| \leq C_1(1 + |z|^k)$ ,  $\forall (t, z, v) \in [0, T] \times E \times \bar{B}(0, R)$ ,
  - (c)  $\exists l, C_2 \geq 0$  such that  $|L_z(t, z, v)| \leq C_2(1 + |z|^l)$ ,  $\forall (t, z, v) \in [0, T] \times E \times \bar{B}(0, R)$ .
- (A.4)

For  $(t, z) \in [0, T] \times E$  fixed, we consider the problem of minimizing

$$J(t, z; \nu) = \mathbb{E} \left[ \int_t^T L(s, Z^{t,z}(s), \nu(s)) ds \right],$$

in  $\nu \in \mathcal{U}_t^2(R)$ . Our goal is here to find the derivative of the function  $u^R : [0, T] \times E \rightarrow \mathbb{R}$ , with respect to the second argument, where  $u^R$  is defined by

$$u^R(t, z) = \inf_{\nu \in \mathcal{U}_t^2(R)} J(t, z; \nu).$$

**Lemma A.1.** *If  $L$  satisfies (A.4),  $\sigma$  is as in (A.1) and  $\nu(\cdot) \in \mathcal{U}_t^2(R)$ , then the derivative  $J_z$  exists and*

$$J_z(t, z; \nu) = \mathbb{E} \left[ \int_t^T L_z(s, Z^{t,z}(s), \nu(s)) ds \right].$$

*Proof.* For  $h \in \mathbb{R} \setminus \{0\}$ ,  $(t, z) \in [0, T] \times E$  and  $\xi \in E$ , we shall prove that

$$\lim_{h \rightarrow 0} \frac{1}{h} (J(t, z + h\xi; \nu) - J(t, z; \nu)) = \mathbb{E} \left[ \int_t^T L_z(s, Z^{t,z}(s), \nu(s)) ds \right] \cdot \xi.$$

Since  $L$  is  $C^{0,1,0}$  we have, for all  $s \in [0, T]$  and  $z_1, z_2, v \in E$ ,

$$L(s, z_1, v) - L(s, z_2, v) = \left( \int_0^1 L_z(s, (1 - \lambda)z_1 + \lambda z_2, v) d\lambda \right) \cdot (z_1 - z_2).$$

This formula and the definition of  $J$  give

$$\begin{aligned} & J(t, z + h\xi; \nu) - J(t, z; \nu) \\ &= \mathbb{E} \left[ \int_t^T (L(s, Z^{t,z+h\xi}(s), \nu(s)) - L(Z^{t,z}(s), \nu(s))) ds \right] = \mathbb{E}[Y_h], \end{aligned} \quad (\text{A.5})$$

where

$$Y_h = \int_t^T \left( \int_0^1 L_z(s, Z^\lambda(t, s), \nu(s)) \cdot (Z^{t,z+h\xi}(s) - Z^{t,z}(s)) d\lambda \right) ds,$$

with  $Z^\lambda(t, s) = (1 - \lambda)Z^{t,z+h\xi}(s) + \lambda Z^{t,z}(s)$ .

In order to study the behavior of  $\mathbb{E}[Y_h]$  for small  $|h|$ , we rewrite  $Y_h$  by using (A.1):

$$Y_h = U_h + V_h, \quad (\text{A.6})$$

where

$$\begin{aligned} U_h &= (h\xi) \cdot \int_t^T \Lambda_h(t, s) ds, \quad V_h = \int_t^T \Lambda_h(t, s) \cdot M_h(t, s) ds, \\ \Lambda_h(t, s) &= \int_0^1 L_z(s, Z^\lambda(t, s), \nu(s)) d\lambda \end{aligned} \quad (\text{A.7})$$

and

$$M_h(t, s) = \int_t^s (\sigma(\tau, Z^{t,z+h\xi}(\tau)) - \sigma(\tau, Z^{t,z}(\tau))) dW_\tau.$$

Due to the Lipschitz property of  $\sigma$

$$\mathbb{E}[(M_h(t, s))^2] = \mathbb{E} \left[ \int_t^s |\sigma(\tau, Z^{t,z+h\xi}(\tau)) - \sigma(\tau, Z^{t,z}(\tau))|^2 d\tau \right] \leq K^2 \mathbb{E} \left[ \int_t^s |Z^{t,z+h\xi}(\tau) - Z^{t,z}(\tau)|^2 d\tau \right].$$

Inequality (A.3) then gives

$$\mathbb{E}[(M_h(t, s))^2] \leq C(s - t)|h\xi|^2,$$

for a  $C \in \mathbb{R}$  independent of  $s, t, z_1, z_2$  and  $v$ . According to (c) of (A.4) we have:

$$|L_z(s, Z^\lambda(t, s), \nu(s))| \leq C_2(1 + |Z^\lambda(s)|^l) \leq C_2(1 + |Z^{t,z}(s)| + |Z^{t,z+h\xi}(s)|)^l.$$

The estimate (A.2) then gives

$$\mathbb{E}[(\Lambda_h(t, s))^2] \leq C(1 + |z| + |z + h\xi|)^{2l},$$

for a  $C \in \mathbb{R}$  independent of  $s, t, z_1, z_2$  and  $v$ . So the Cauchy-Schwartz inequality gives

$$\mathbb{E}[|V_h|] \leq \mathbb{E} \left[ \int_t^T |\Lambda_h(t, s) \cdot M_h(t, s)| ds \right] \leq C|h\xi|(1 + |z| + |z + h\xi|)^l,$$

for a  $C \in \mathbb{R}$  independent of  $s, t, z_1, z_2$  and  $v$ . Since  $M_h(t, \cdot)$  is a martingale restricted to  $[t, T]$ , it now follows from (A.7) and the Fubini theorem that

$$\mathbb{E}[V_h] = 0. \quad (\text{A.8})$$

For all  $\varepsilon > 0$ , we obtain by Markov's inequality and by (A.3), with  $q = 2$ , that

$$\mathbb{Q} \left( \sup_{t \leq s \leq T} |Z^{t, z+h\xi}(s) - Z^{t, z}(s)| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^2} \mathbb{E} \left[ \sup_{t \leq s \leq T} |Z^{t, z+h\xi}(s) - Z^{t, z}(s)|^2 \right] \leq \frac{Ch^2|\xi|}{\varepsilon^2},$$

where  $C \in \mathbb{R}$  is independent of  $t, z_1, z_2$  and  $v$ . So  $\sup_{t \leq s \leq T} |Z^{t, z+h\xi}(s) - Z^{t, z}(s)|$  converges to 0 in probability, when  $h \rightarrow 0$ . By the continuity of the function

$$[0, T] \times E \times E \times E \ni (s, z_1, z_2, z) \mapsto \int_0^1 L_z(s, (1-\lambda)z_1 + \lambda z_2, v) d\lambda - L_z(s, z, v)$$

it then follows that also  $\sup_{t \leq s \leq T} |\Lambda_h(t, s) - L_z(s, Z^{t, z}(s), \nu(s))|$  converges to 0 in probability, as  $h \searrow 0$ . This is then also the case for  $\int_t^T (\Lambda_h(t, s) - L_z(s, Z^{t, z}(s), \nu(s))) ds$ .

By (c) of (A.4) and (A.2) it follows that the family

$$\left\{ \int_t^T (\Lambda_h(t, s) - L_z(s, Z^{t, z}(s), \nu(s))) ds : |h| \leq 1 \right\}$$

is uniformly integrable. This gives that

$$\lim_{h \rightarrow 0} \mathbb{E} \left[ \left| \int_t^T (\Lambda_h(t, s) - L_z(s, Z^{t, z}(s), \nu(s))) ds \right| \right] = 0,$$

which, with (A.6) and (A.8), shows that

$$\lim_{h \rightarrow 0} \mathbb{E} \left[ \left| \frac{1}{h} Y_h - \xi \cdot L_z(s, Z^{t, z}(s), \nu(s)) ds \right| \right] = 0.$$

□

Given  $(t, z) \in [0, T] \times E$ , let  $\nu_*^R(t, z)$  be the unique element in  $\mathcal{U}_t^2(R)$  for which  $\nu \mapsto J(t, z; \nu)$  takes its minimum on  $\mathcal{U}_t^2(R)$ . Now let  $Z_*^{t, z} = (Z_*^{t, z}(s))_{t \leq s \leq T}$  be the solution to

$$Z_*^{t, z}(s) = z + \int_t^s \nu_*^R(u, Z_*^{t, z}(u)) du + \int_t^s \sigma(u, Z_*^{t, z}(u)) dW_u, \quad t \leq s \leq T. \quad (\text{A.9})$$

**Corollary A.2.** *For every  $(t, z) \in [0, T] \times E$ ,*

$$u_z^R(t, z) = \mathbb{E} \left[ \int_t^T L_z(s, Z_*^{t, z}(s), \nu_*^R(s, Z_*^{t, z}(s))) ds \right], \quad (\text{A.10})$$

where  $Z_*^{t, z}$  is the solution to (A.9).

*Proof.* Since  $\nu_*^R$  is an optimal Markov control policy,

$$u^R(t, z) = J(t, z; \nu_*^R(t, z)).$$

Given  $(t, z) \in [0, T] \times E$ ,  $h > 0$  and  $\xi \in E$  we obtain

$$\begin{aligned} \frac{1}{h}(u^R(t, z + h\xi) - u^R(t, z)) &= \frac{1}{h} \left( \inf_{\nu \in \mathcal{U}_t^2(R)} J(t, z + h\xi; \nu) - J(t, z; \nu_*^R(t, z)) \right) \\ &\leq \frac{1}{h} (J(t, z + h\xi; \nu_*^R(t, z)) - J(t, z; \nu_*^R(t, z))); \end{aligned} \quad (\text{A.11})$$

and the right side tends to  $J_z(t, z; \nu_*^R(t, z)) \cdot \xi$  as  $h \searrow 0$ . Therefore, by precedent lemma,

$$u_z^R(t, z) \cdot \xi \leq \int_t^T L_z(s, Z_*^{t,z}(s), \nu_*^R(s, Z_*^{t,z}(s))) ds \cdot \xi.$$

This holds for all directions  $\xi$ , in particular with  $\xi$  replaced by  $-\xi$ , which gives (A.10).  $\square$

## References

- [1] Berdjane, B. and Pergamenshchikov, S.: *Optimal consumption and investment for markets with random coefficients*, Finance Stoch. **17**, 419–446 (2013)
- [2] Bergomi, L. and Guyon, J.: *The Smile in Stochastic Volatility Models*, Preprint 2011, Available at <http://ssrn.com/abstract=1967470>
- [3] Björk, T. and Svensson, L.: On the Existence of Finite Dimensional Realizations for Nonlinear Forward Rate Models, Mathematical Finance, **11**, 205–243 (2001).
- [4] Buehler, H.: *Consistent variance curve models*, Finance Stoch. **10**, 178–203 (2006)
- [5] Castañeda-Leyva, N. and Hernández-Hernández, D.: *Optimal consumption investment problems in incomplete markets with stochastic coefficients*, SIAM J. Control Optim. **44**, 1322–1344 (2005)
- [6] Delong, L. and Klüppelberg, C.: *Optimal investment and consumption in a Black-Scholes market with Lévy-driven stochastic coefficients*, Ann. Appl. Probab. **18**, 879–908 (2008)
- [7] Ekeland, I. and Taflin, E.: *A theory of bond portfolios*, The Annals of Applied Probability, **15**, 1260–1305 (2005).
- [8] Filipovic, D. and Teichmann, J.: *On the geometry of the term structure of interest rates*, Proc. R. Soc. Lond. Ser. A **460**, 129–167 (2004)



- [9] Fleming, W. and Rishel R.: *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York (1975).
- [10] Fleming, W. and Soner, H.M.: *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York (1993).
- [11] Krylov, N.V.: *Controlled Diffusion Processes*, Springer-Verlag, New York (1980)
- [12] Lindberg, C.: *Portfolio optimization and a factor model in a stochastic volatility market*, Stochastics **78**, 259–279 (2006)
- [13] Merton, R.: *Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time case*, Rev. Economics and Stat. **51**, 247–257 (1969).
- [14] Merton, R.: *Optimum Consumption and Portfolio Rules in a Continuous Time Model*, Jour. Economic Theory, **3**, 373–413 (1971).
- [15] Pham H.: *Smooth Solutions to Optimal Investment Models with Stochastic Volatilities and Portfolio Constraints*, Appl. Math. Optim. **46**, 55–78 (2002).
- [16] Revuz, D. and Yor, M.: *Continuous Martingales and Brownian Motion*, Grundlehren der mathematischen Wissenschaften, Band 293, Springer-Verlag 2004.
- [17] Touzi, N.: *Optimal Stochastic Control, Stochastic Target Problems, and Backward SDE*, Springer 2013.
- [18] Zariphopoulou, T.: *A solution approach to valuation with unhedgeable risks*, Finance Stoch. **5**, 61–82 (2001)